MATH 306 Workshop

1. Define what it means for a list of vectors in V to be **linearly independent**.

2. Define what it means for a list of vectors in V to be a **basis** of V.

3. State the **Fundamental Theorem of Linear Maps**.

4. Let $T \in \mathcal{L}(V)$, define what is means for a subspace U of V to be **invariant** under T.

5. State the theorem about **diagonalizability**.

6. True or False (if false, explain or give a counterexample):

   a. $W = \{(x, 1, x) : x \in \mathbf{F}\}$ is a subspace of $\mathbf{F}^3$.

   b. (1, 0, 0), (0, 2, 0), (0, 0, 3), (3, 4, 8) spans $\mathbf{R}^3$.

   c. Dim $\mathbf{F}^{3,4} = 7$

   d. U and W are subspaces of $\mathbf{F}^{10}$ with $\dim(U) = 8$, $\dim(W) = 6$, and $\dim(U \cap W) = 2$

   e. $T(x) = \sin(x)$ is a linear map from $\mathbf{R}$ to $\mathbf{R}$.

   f. If $T \in L(V, W)$ is both injective and surjective, then it is invertible.

   g. Let $T \in L(V)$, null T is invariant under T

h. Let $T \in L(V)$ where V is a finite-dimensional complex vector space. Given T has two eigenvalues: $\lambda = 1$ with corresponding eigenvector v_1 and $\lambda = 3$ with corresponding eigenvectors v_2 and v_3. Then the trace of T is 4.

i. Every operator T on a non-zero, complex vector space has an eigenvalue.

j. Every operator T on a non-zero, finite-dimensional vector space has an eigenvalue.

k. Suppose V is a finite-dimensional non-trivial complex vector space, then every operator in V has an upper-triangular matrix.

7. $Suppose\ T \in \mathcal{L}(V)\ and\ (T-2I)*(T-3I)*(T-4I)=0.$

   $Suppose\ \lambda\ is\ an\ eigenvalue\ of\ T. Prove\ that\ \lambda = 2\ or\ \lambda = 3\ or\ \lambda = 4.$

8. Let V = $\mathbf{R}^{[0,\,1]}$, and let U = {f ∈ V : f(0.5) = 0}.

   Prove or disprove that U is a subspace of V.

9. Prove that the set of all vectors $(x_1, \ldots, x_n)$ in $\mathbf{R}^n$ that satisfy $x_1 + \ldots + x_n = 0$ is a subspace of $\mathbf{R}^n$. And find a basis for this subspace.

10. Let U be the subspace of $\mathbf{F}^5$ defined by:

$$U = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_2 = x_1, x_3 = 2x_4 - 3x_5 \}$$

a. Find a basis for U and prove that your answer is a basis.

b. Extend the basis in part (a) to a basis of $\mathbf{F}^5$.

c. Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

11. Let T: $\mathbf{F}^2 \rightarrow \mathbf{F}^3$ be defined by $T(x_1, x_2) = (x_1 + x_2, x_1, 0)$. Show T is a linear map.

12. Let T: $\mathbf{F}^3 \rightarrow \mathbf{F}^2$ be defined by $T(x, y, z) = (x - y, 2z)$. Find the null T and what is a basis for null T.

13. Let $V = F^{2,2}$. Define $T \in L(V)$ by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$$

Find all eigenvalues and eigenvectors of T.

14. Define $T \in L(\mathbf{C}^2)$ by $T(z, w) = (4z + 3w, 3z + 4w)$. The vectors $(1, 1)$ and $(1, -1)$ are eigenvectors of T. Find all of the eigenvalues of T and explain why your answer is complete.

15. Let T be defined on a vector space of dimension 10 has eigenvalues 3, 9, and 11 and no others. If dim E(3,T) = 2, dim E(9,T) = 5, and dim E (11, T) = 2, is T diagonalizable?

16. $Given\ \lambda\ is\ an\ eigenvalue\ of\ T \in \mathcal{L}(V), prove\ that\ T - \lambda I\ is\ not\ invertible.$

17. $Suppose\ T \in \mathcal{L}(V)\ with\ T^2 = I\ and - 1\ is\ not\ an\ eigenvalue\ of\ T. Prove\ that$ $T\ is\ the\ identity\ map.$

18. Given $det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 7$, calculate the following:

a. $det\begin{pmatrix} a & b & c \\ a & b & c \\ g & h & i \end{pmatrix} =$

b. $det\begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix} =$

19. Given an operator on V is invertible, what can you say about its injectivity, surjectivity, and determinant?

Good Luck on the FINAL!